Heptavalent symmetric graphs of order $12p$

Song-Tao Guo
School of Mathematics and Statistics
Henan University of Science and Technology
Luoyang, 471023
P.R. China
gsongtao@gmail.com

Abstract. A graph is symmetric if its automorphism group acts transitively on the set of arcs of the graph. In this paper, we classify connected heptavalent symmetric graphs of order $12p$ for each prime p. As a result, there are eleven sporadic such graphs: one for $p = 2$, one for $p = 3$ and nine for $p = 13$.

Keywords: symmetric graph, s-transitive graph, coset graph.

1. Introduction

Throughout this paper graphs are assumed to be finite, simple, connected and undirected. For group-theoretic concepts or graph-theoretic terms not defined here we refer the reader to [24, 27] or [1, 2], respectively. Let G be a permutation group on a set Ω and $v \in \Omega$. Denote by G_v the stabilizer of v in G, that is, the subgroup of G fixing the point v. We say that G is *semiregular* on Ω if $G_v = 1$ for every $v \in \Omega$ and *regular* if G is transitive and semiregular.

For a graph X, denote by $V(X)$, $E(X)$ and $\mathrm{Aut}(X)$ its vertex set, its edge set and its full automorphism group, respectively. A graph X is said to be *G-vertex-transitive* if $G \leq \mathrm{Aut}(X)$ acts transitively on $V(X)$. X is simply called *vertex-transitive* if it is $\mathrm{Aut}(X)$-vertex-transitive. An *s-arc* in a graph is an ordered $(s+1)$-tuple $(v_0, v_1, \cdots, v_{s-1}, v_s)$ of vertices of the graph X such that v_{i-1} is adjacent to v_i for $1 \leq i \leq s$, and $v_{i-1} \neq v_{i+1}$ for $1 \leq i \leq s-1$. In particular, a 1-arc is just an arc and a 0-arc is a vertex. For a subgroup $G \leq \mathrm{Aut}(X)$, a graph X is said to be *(G, s)-arc-transitive* or *(G, s)-regular* if G is transitive or regular on the set of s-arcs in X, respectively. A (G, s)-arc-transitive graph is said to be *(G, s)-transitive* if it is not $(G, s+1)$-arc-transitive. In particular, a $(G, 1)$-arc-transitive graph is called *G-symmetric*. A graph X is simply called *s-arc-transitive*, *s-regular* or *s-transitive* if it is $(\mathrm{Aut}(X), s)$-arc-transitive, $(\mathrm{Aut}(X), s)$-regular or $(\mathrm{Aut}(X), s)$-transitive, respectively.

As we all known that the structure of the vertex stabilizers of symmetric graphs is very useful to classify such graphs, and this structure of the cubic or tetravalent case was given by Miller [20] and Potočnik [23]. Thus, classifying symmetric graphs with valency 3 or 4 has received considerable attention and a lot of results have been achieved, see [8, 30, 31]. Guo [12] determined the exact structure of pentavalent case. Following this structure, a series of pentavalent symmetric graphs was classified in [17, 21, 22, 28, 29]. Recently, Guo [13] gave the exact structure of heptavalent case and determined heptavalent symmetric graph of order $6p$ in [10]. Thus, as an application, we classify connected heptavalent symmetric graphs of order $12p$ for each prime p in this paper.

2. Preliminary results

Let X be a connected G-symmetric graph with $G \leq \mathrm{Aut}(X)$, and let N be a normal subgroup of G. The *quotient graph* X_N of X relative to N is defined as the graph with vertices the orbits of N on $V(X)$ and with two orbits adjacent if there is an edge in X between those two orbits. In view of [18, Theorem 9], we have the following:

Proposition 2.1. *Let X be a connected heptavalent G-symmetric graph with $G \leq \mathrm{Aut}(X)$, and let N be a normal subgroup of G. Then one of the following holds:*

(1) *N is transitive on $V(X)$;*

(2) *X is bipartite and N is transitive on each part of the bipartition;*

(3) *N has $r \geq 3$ orbits on $V(X)$, N acts semiregularly on $V(X)$, the quotient graph X_N is a connected heptavalent G/N-symmetric graph.*

The following proposition characterizes the vertex stabilizers of connected heptavalent s-transitive graphs (see [13, Theorem 1.1]).

Proposition 2.2. *Let X be a connected heptavalent (G, s)-transitive graph for some $G \leq \mathrm{Aut}(X)$ and $s \geq 1$. Let $v \in V(X)$. Then $s \leq 3$ and one of the following holds:*

(1) *For $s = 1$, $G_v \cong \mathbb{Z}_7$, D_{14}, F_{21}, D_{28}, $F_{21} \times \mathbb{Z}_3$;*

(2) *For $s = 2$, $G_v \cong F_{42}$, $F_{42} \times \mathbb{Z}_2$, $F_{42} \times \mathbb{Z}_3$, $\mathrm{PSL}(3,2)$, A_7, S_7, $\mathbb{Z}_2^3 \rtimes \mathrm{SL}(3,2)$ or $\mathbb{Z}_2^4 \rtimes \mathrm{SL}(3,2)$;*

(3) *For $s = 3$, $G_v \cong F_{42} \times \mathbb{Z}_6$, $\mathrm{PSL}(3,2) \times \mathrm{S}_4$, $\mathrm{A}_7 \times \mathrm{A}_6$, $\mathrm{S}_7 \times \mathrm{S}_6$, $(\mathrm{A}_7 \times \mathrm{A}_6) \rtimes \mathbb{Z}_2$, $\mathbb{Z}_2^6 \rtimes (\mathrm{SL}(2,2) \times \mathrm{SL}(3,2))$ or $[2^{20}] \rtimes (\mathrm{SL}(2,2) \times \mathrm{SL}(3,2))$.*

From [9, pp.12-14], [26, Theorem 2] and [16, Theorem A], we may obtain the following proposition by checking the orders of non-abelian simple groups:

Proposition 2.3. *Let p be a prime, and let G be a non-abelian simple group of order $|G| \mid (2^{26} \cdot 3^5 \cdot 5^2 \cdot 7 \cdot p)$. Then G has 3-prime factor, 4-prime factor or 5-prime factor, and is one of the following groups:*

Table 1: **Non-abelian simple $\{2, 3, 5, 7, p\}-$groups**

3-prime factor					
G	Order	G	Order	G	Order
A_5	$2^2{\cdot}3{\cdot}5$	PSL(2, 8)	$2^3{\cdot}3^2{\cdot}7$	PSU(4, 2)	$2^6{\cdot}3^4{\cdot}5$
A_6	$2^3{\cdot}3^2{\cdot}5$	PSL(2, 17)	$2^4{\cdot}3^2{\cdot}17$	PSU(3, 3)	$2^5{\cdot}3^3{\cdot}7$
PSL(2, 7)	$2^3{\cdot}3{\cdot}7$	PSL(3, 3)	$2^4{\cdot}3^3{\cdot}13$		
4-prime factor					
G	Order	G	Order	G	Order
A_7	$2^3{\cdot}3^2{\cdot}5{\cdot}7$	PSL(2, 27)	$2^2{\cdot}3^3{\cdot}7{\cdot}13$	PSU(5, 2)	$2^{10}{\cdot}3^5{\cdot}5{\cdot}11$
A_8	$2^6{\cdot}3^2{\cdot}5{\cdot}7$	PSL(2, 31)	$2^5{\cdot}3{\cdot}5{\cdot}31$	PSp(4, 4)	$2^8{\cdot}3^2{\cdot}5^2{\cdot}17$
A_9	$2^6{\cdot}3^4{\cdot}5{\cdot}7$	PSL(2, 49)	$2^4{\cdot}3{\cdot}5^2{\cdot}7^2$	PSp(6, 2)	$2^9{\cdot}3^4{\cdot}5{\cdot}7$
A_{10}	$2^7{\cdot}3^4{\cdot}5^2{\cdot}7$	PSL(2, 81)	$2^4{\cdot}3^4{\cdot}5{\cdot}41$	M_{11}	$2^4{\cdot}3^2{\cdot}5{\cdot}11$
PSL(2, 11)	$2^2{\cdot}3{\cdot}5{\cdot}11$	PSL(2, 127)	$2^7{\cdot}3^2{\cdot}7{\cdot}127$	M_{12}	$2^6{\cdot}3^3{\cdot}5{\cdot}11$
PSL(2, 13)	$2^2{\cdot}3{\cdot}7{\cdot}13$	PSL(3, 4)	$2^6{\cdot}3^2{\cdot}5{\cdot}7$	J_2	$2^7{\cdot}3^3{\cdot}5^2{\cdot}7$
PSL(2, 16)	$2^4{\cdot}3{\cdot}5{\cdot}17$	PSU(3, 4)	$2^6{\cdot}3{\cdot}5^2{\cdot}13$	$P\Omega^+(8, 2)$	$2^{12}{\cdot}3^5{\cdot}5^2{\cdot}7$
PSL(2, 19)	$2^2{\cdot}3^2{\cdot}5{\cdot}19$	PSU(3, 5)	$2^4{\cdot}3^2{\cdot}5^3{\cdot}7$	Sz(8)	$2^6{\cdot}5{\cdot}7{\cdot}13$
PSL(2, 25)	$2^3{\cdot}3{\cdot}5^2{\cdot}13$	PSU(3, 8)	$2^9{\cdot}3^4{\cdot}7{\cdot}19$	${}^2F_4(2)'$	$2^{11}{\cdot}3^3{\cdot}5^2{\cdot}13$
5-prime factor					
G	Order	G	Order	G	Order
A_{11}	$2^7{\cdot}3^4{\cdot}5^2{\cdot}7{\cdot}11$	PSL(2, 449)	$2^6{\cdot}3^2{\cdot}5^2{\cdot}7{\cdot}449$	PSp(8, 2)	$2^{16}{\cdot}3^5{\cdot}5^2{\cdot}7{\cdot}17$
A_{12}	$2^9{\cdot}3^5{\cdot}5^2{\cdot}7{\cdot}11$	PSL(2, 2^6)	$2^6{\cdot}3^2{\cdot}5{\cdot}7{\cdot}13$	M_{22}	$2^7{\cdot}3^2{\cdot}5{\cdot}7{\cdot}11$
PSL(2, 29)	$2^2{\cdot}3{\cdot}5{\cdot}7{\cdot}29$	PSL(4, 4)	$2^{12}{\cdot}3^4{\cdot}5^2{\cdot}7{\cdot}17$	$P\Omega^-(8, 2)$	$2^{12}{\cdot}3^4{\cdot}5{\cdot}7{\cdot}17$
PSL(2, 41)	$2^3{\cdot}3{\cdot}5{\cdot}7{\cdot}41$	PSL(5, 2)	$2^{10}{\cdot}3^2{\cdot}5{\cdot}7{\cdot}31$	$G_2(4)$	$2^{12}{\cdot}3^3{\cdot}5^2{\cdot}7{\cdot}13$
PSL(2, 71)	$2^3{\cdot}3^2{\cdot}5{\cdot}7{\cdot}71$				

In view of [14, Theorem 3.1], we have the classification of connected heptavalent symmetric graphs of order $4p$ for a prime p.

Proposition 2.4. *Let X be connected heptavalent symmetric graph of order $4p$ with p a prime. Then X is isomorphic to K_8.*

Next we construct some heptavalent symmetric graphs of order $6p$ with p a prime. In order to construct some heptavalent symmetric graphs, we need to introduce the so called coset graph (see [20, 25]) constructed from a finite group G relative to a subgroup H of G and a union D of some double cosets of H in G such that $D^{-1} = D$. The *coset graph* $\mathrm{Cos}(G, H, D)$ of G with respect to H and D is defined to have vertex set $[G : H]$, the set of right cosets of H in G, and edge set $\{\{Hg, Hdg\} \mid g \in G, d \in D\}$. The graph $\mathrm{Cos}(G, H, D)$ has valency $|D|/|H|$ and is connected if and only if D generates the group G. The action of G on $V(\mathrm{Cos}(G, H, D))$ by right multiplication induces a vertex-transitive automorphism group, which is arc-transitive if and only if D is a single double coset. Moreover, this action is faithful if and only if $H_G = 1$, where H_G is the largest normal subgroup of G in H. Clearly, $\mathrm{Cos}(G, H, D) \cong \mathrm{Cos}(G, H^\alpha, D^\alpha)$ for every $\alpha \in \mathrm{Aut}(G)$. For more details regarding coset graphs, see, for example, [7, 18, 25].

By Atlas [6], S_8 has a maximal subgroup $K \cong \mathbb{Z}_2^4 \rtimes S_4$ and A_8 has a maximal subgroup $H \cong \mathbb{Z}_2^3 \rtimes \mathrm{PSL}(3, 2)$ such that $K \cap H \cong \mathbb{Z}_2^3 \rtimes S_4$. Take an element g

of order 2 in $K\backslash(K\cap H)$. Then define the coset graph:

$$\mathcal{C}_{30}=\mathrm{Cos}(\mathrm{S}_8,H,HgH).$$

Let $G\cong\mathrm{PSL}(2,13)$ as a permutation group acting naturally on 14 points, and take the following four elements:

$a=(1,13,2,12,9,5,14)(3,4,7,10,11,6,8)$,
$b=(2,5)(3,7)(6,11)(8,10)(9,12)(13,14)$,
$x=(1,4)(3,10)(6,14)(7,8)(9,12)(11,13)$,
$y=(1,4)(2,6)(5,11)(8,9)(10,12)(13,14)$.

Then $H=\langle a,b\rangle\cong D_{14}$. Define the following two coset graph:

$$\mathcal{C}^1_{78}=\mathrm{Cos}(G,H,HxH),\qquad \mathcal{C}^2_{78}=\mathrm{Cos}(G,H,HyH).$$

We can get the classification of heptavalent symmetric graphs of order $6p$ with p a prime from [10, Theorem 3.1].

Proposition 2.5. *Let X be a connected heptavalent symmetric graph of order $6p$ with p a prime. Then X is isomorphic to $\mathcal{C}_{30}$, $\mathcal{C}^1_{78}$ or $\mathcal{C}^2_{78}$.*

Let $G\cong\mathrm{PSL}(2,8)$. Then by Atlas [6], G has a maximal subgroup $H\cong D_{14}$ and a Sylow 2-subgroup $P\cong\mathbb{Z}_2^3$ such that $P\cap H\cong\mathbb{Z}_2$. Take $g\in P\backslash H$. Then $g^2=1$, $H\cap H^g\cong\mathbb{Z}_2$ and $\langle H,g\rangle=G$. Define the coset graph:

$$\mathcal{C}_{36}=\mathrm{Cos}(G,H,HgH).$$

From [11, Theorem 3.1], we have the classification of connected heptavalent symmetric graphs of order 36.

Proposition 2.6. *There is only one connected heptavalent symmetric graph of order* 36*, that is, $\mathcal{C}_{36}$.*

Now we construct some heptavalent symmetric graphs of order $12p$. If $p=2$, then by [5], there is only one such graph of order 24, and we define this graph as follows:

Construction 2.7. *Set $a=(3,5)(4,7)(6,8)$, $b=(1,2)(4,8)(6,7)$ and $c=(1,4,8,3,5,6,7)$. Then $G=\langle a,b,c\rangle\cong\mathrm{PGL}(2,7)$. Let $H=\langle a,c\rangle$. Clearly, $H\cong D_{14}$ and b centralizes a. Define the coset graph:*

$$\mathcal{C}_{24}=\mathrm{Cos}(G,H,HbH).$$

Then $\mathcal{C}_{24}$ is a connected heptavalent symmetric graph of order 24*. By Magma [3],* $\mathrm{Aut}(\mathcal{C}_{24})\cong\mathrm{PGL}(2,7)$ *and any connected heptavalent symmetric graph of order* 24 *admitting* $\mathrm{PGL}(2,7)$ *as an arc-transitive automorphism group is isomorphic to $\mathcal{C}_{24}$.*

By using the isomorphisms of coset graph and calculation of Magma [3], we have that there are nine non-isomorphic heptavalent symmetric graphs of order 156 with $\mathrm{PSL}(2,13)$ as an automorphism group.

Construction 2.8. *Take the following six elements in* S_{16}*:*

$$\begin{aligned}
a &= (3,13,11,9,7,5)(4,14,12,10,8,6),\\
b &= (1,2,9)(3,8,10)(4,5,12)(6,13,14),\\
c &= (15,16),\\
f_1 &= (2,10)(3,13)(4,11)(5,14)(6,7)(8,12),\\
f_2 &= (2,10)(3,13)(4,11)(5,14)(6,7)(8,12)(15,16),\\
f_3 &= (1,10,8,5,14,12,2)(3,6,11,4,7,13,9).
\end{aligned}$$

Then $G = \langle a,b,c\rangle \cong \mathrm{PSL}(2,13)\times \mathbb{Z}_2$*,* $H_1 = \langle f_1, f_3\rangle \cong D_{14}$ *and* $H_2 = \langle f_2, f_3\rangle \cong D_{14}$*. Take five elements in* G*:*

$$\begin{aligned}
x_1 &= (3,9)(4,10)(5,11)(6,12)(7,13)(8,14)(15,16),\\
x_2 &= (2,6)(3,14)(4,8)(7,12)(9,11)(10,13)(15,16),\\
x_3 &= (2,9)(3,10)(4,13)(5,6)(7,11)(12,14)(15,16),\\
x_4 &= (1,9)(2,6)(3,13)(5,8)(7,10)(12,14)(15,16),\\
x_5 &= (1,9)(2,4)(3,14)(5,13)(8,12)(10,11)(15,16).
\end{aligned}$$

Define the following five coset graphs:

$$\begin{aligned}
\mathcal{SG}^i_{156} &= \mathrm{Cos}(G, H_1, H_1x_iH_1), i = 1,2.\\
\mathcal{SG}^i_{156} &= \mathrm{Cos}(G, H_2, H_2x_iH_2),\ i = 3,4,5.
\end{aligned}$$

By Magma [3], $\mathrm{Aut}(\mathcal{SG}^1_{156}) \cong \mathrm{PSL}(2,13)\times\mathbb{Z}_2$*,* $\mathrm{Aut}(\mathcal{SG}^2_{156}) \cong \mathrm{PGL}(2,13)\times\mathbb{Z}_2$ *and* $\mathrm{Aut}(\mathcal{SG}^i_{156}) \cong \mathrm{PSL}(2,13)\times\mathbb{Z}_2$ *for* $i = 3,4,5$*. In particular,* $\mathcal{SG}^1_{156}$ *and* $\mathcal{SG}^2_{156}$ *are bipartite.*

Take the following eight elements:

$$\begin{aligned}
d &= (1,2)(3,6)(4,5)(7,14)(8,13)(9,12)(10,11),\\
e &= (1,4)(2,5)(3,8)(6,13)(7,10)(11,14),\\
f &= (1,9,4,14,3,8,11)(2,10,13,6,7,5,12),\\
y_1 &= (3,9)(4,10)(5,11)(6,12)(7,13)(8,14)\\
y_2 &= (2,11)(3,6)(4,14)(5,12)(7,8)(9,13),\\
y_3 &= (2,13)(3,11)(4,6)(5,8)(7,14)(9,10),\\
y_4 &= (2,9,5,12)(3,14,7,13)(6,10,11,8),\\
y_5 &= (2,13,6,10)(3,4,14,8)(7,9,12,11),
\end{aligned}$$

Set $T = \langle a,b,d\rangle \cong \mathrm{PGL}(2,13)$*,* $K_1 = \langle d,f\rangle \cong D_{14}$ *and* $K_2 = \langle e,f\rangle \cong D_{14}$*. Define the following coset graphs:*

$$\begin{aligned}
\mathcal{CG}^i_{156} &= \mathrm{Cos}(T, K_1, K_1y_iK_1), i = 1,2,3.\\
\mathcal{CG}^i_{156} &= \mathrm{Cos}(T, K_2, K_2y_iK_2),\ i = 4,5.
\end{aligned}$$

By Magma [3], $\mathcal{CG}_{156}^5 \cong \mathcal{SG}_{156}^2$, *and* $\mathrm{Aut}(\mathcal{CG}_{156}^i) \cong \mathrm{PGL}(2,13)$ *for* $i = 1,2,3,4$. *In particular,* $\mathcal{CG}_{156}^4$ *is bipartite.*

We use the same notations as above to state the following lemma.

Lemma 2.9. *Let* X *be a connected heptavalent symmetric graph of order* 156 *and* $A = \mathrm{Aut}(X)$. *Then*

(1) *If* A *has an arc-transitive subgroup isomorphic to* G, *then* $X \cong \mathcal{SG}_{156}^i$ *for* $i = 1,2,3,4,5$;

(2) *If* A *has an arc-transitive subgroup isomorphic to* T, *Then* $X \cong \mathcal{CG}_{156}^i$ *for* $i = 1,2,3,4,5$;

Proof. Assume that A has an arc-transitive subgroup G. Then $X \cong \mathrm{Cos}(G, G_v, G_vgG_v)$ with $G_v \cong D_{14}$, $\langle G_v, g\rangle = G$ and $|G_v : G_v^g \cap G_v| = 7$. In particular, g can be chosen as a 2-element. By Magma [3], G has two conjugacy classes of subgroups isomorphic to D_{14} with H_1 and H_2 as their representatives.

Suppose that $G_v = H_1$. Then by Magma [3], g has 42 choices, denoted this set by U. Note that $N_G(G_v) \cong D_{28}$ by Atlas [6]. Thus, again by Magma [3], $N_G(G_v)$ acting on U has 6 orbits. Clearly, the coset graphs formed by the elements in the same orbit are isomorphic each other. Thus, we obtain six coset graphs. By Magma [3], these six representatives of 6 orbits form two coset graphs, and $X \cong \mathcal{SG}_{156}^i$ with $i = 1,2$.

Suppose that $G_v = H_2$. Then by Magma [3], g has 84 choices, denoted this set by V. Since $N_G(G_v) \cong D_{28}$, we have that $N_G(G_v)$ acting on V has 12 orbits by Magma [3]. All the representatives of 12 orbits form three coset graphs, and $X \cong \mathcal{SG}_{156}^i$ with $i = 3,4,5$.

Assume that A has an arc-transitive subgroup T. Then $X \cong \mathrm{Cos}(T, T_v, T_vtT_v)$ with $T_v \cong D_{14}$, $\langle T_v, t\rangle = T$ and $|T_v : T_v^t \cap T_v| = 7$. In particular, t can be chosen as a 2-element. By Atlas [6], T has two conjugacy classes of subgroups isomorphic to D_{14} with K_1 and K_2 as their representatives.

Suppose that $T_v = K_1$. Then by Magma [3], t has 112 choices, and $N_T(T_v) \cong D_{28}$ acting on these elements has seven orbits. These seven orbits form seven coset graphs and by Magma [3], there are 3 non-isomorphic graphs, that is, $X \cong \mathcal{CG}_{156}^i$ with $i = 1,2,3$.

Suppose that $T_v = K_2$. Then by Magma [3], t has 98 choices, and $N_T(T_v) \cong D_{28}$ acting on these elements has five orbits. These five orbits form five coset graphs and by Magma [3], there are 2 non-isomorphic graphs, that is, $X \cong \mathcal{CG}_{156}^i$ with $i = 4,5$. $\square$

3. Classification

This section is devoted to classify connected heptavalent symmetric graphs of order $12p$ for each prime p.

Theorem 3.1. *Any connected heptavalent symmetric graph of order* $12p$ *with* p *a prime is isomorphic to* $\mathcal{C}_{24}$, $\mathcal{C}_{36}$, $\mathcal{SG}^i_{156}$ *with* $i = 1, 2, 3, 4, 5$, *or* $\mathcal{CG}^j_{156}$ *with* $j = 1, 2, 3, 4$.

Proof. Let X be a connected heptavalent symmetric graph of order $12p$ and $A = \mathrm{Aut}(X)$. If $p = 2$, then by [5] and Construction 2.7, $X \cong \mathcal{C}_{24}$ and $A \cong \mathrm{PGL}(2, 7)$. If $p = 3$, then by Proposition 2.6, $X \cong \mathcal{C}_{36}$. Thus, in the following we assume that $p \geq 5$. Take $v \in V(X)$. Then by Proposition 2.2, $|A_v| \,\big|\, 2^{24}{\cdot}3^4{\cdot}5^2{\cdot}7$ and hence $|A| \,\big|\, 2^{26}{\cdot}3^5{\cdot}5^2{\cdot}7{\cdot}p$. We separate the proof into two cases: A has a solvable minimal normal subgroup; A has no solvable minimal normal subgroup.

Case 1: A has a solvable minimal normal subgroup.

Let N be a solvable minimal normal subgroup of A. Then $|N| \,\big|\, 2^{26}{\cdot}3^5{\cdot}5^2{\cdot}7{\cdot}p$, and N is elementary abelian. Thus, $N \cong \mathbb{Z}_q^k$ with $q = 2, 3, 5, 7$ or p and k a positive integer. By Proposition 2.1, N is semiregular and X_N is also a connected heptavalent A/N-symmetric graph. It follows that $|N| \,\big|\, 12p$ and $N \cong \mathbb{Z}_2$, $\mathbb{Z}_2^2$, $\mathbb{Z}_3$ or $\mathbb{Z}_p$. Note that there is no connected heptavalent regular graph of odd order. Thus, $N \ncong \mathbb{Z}_2^2$. Since there is no connected heptavalent regular graph of order 12 by [19] or [5], we have that $N \ncong \mathbb{Z}_p$. This forces that $N \cong \mathbb{Z}_2$ or $\mathbb{Z}_3$ and X_N is a heptavalent symmetric graph of order $6p$ or $4p$. By Propositions 2.4 and 2.5, $X_N \cong K_8$, $\mathcal{C}_{30}$, $\mathcal{C}^1_{78}$ or $\mathcal{C}^2_{78}$. If $X_N \cong K_8$, then $N \cong \mathbb{Z}_3$ and $p = 2$, contrary to our assumption. Thus, $N \cong \mathbb{Z}_2$ and $X_N \cong \mathcal{C}_{30}$, $\mathcal{C}^1_{78}$ or $\mathcal{C}^2_{78}$.

Assume that $X_N \cong \mathcal{C}_{30}$. Then $p = 5$. Since $\mathrm{Aut}(\mathcal{C}_{30}) \cong \mathrm{S}_8$ by Proposition 2.5, we have that $A/N \lesssim \mathrm{S}_8$. By Magma [3], $\mathrm{Aut}(\mathcal{C}_{30})$ has a minimal arc-transitive subgroup isomorphic to S_7. It follows that A/N has an arc-transitive subgroup $M/N \cong \mathrm{S}_7$. Note that $\mathcal{C}_{30}$ is bipartite. Set $H/N \cong \mathrm{A}_7$. Then H/N has two orbits on $V(X_N)$ and hence $H_v \cong \mathrm{PSL}(2, 7)$. Since $N \leq C_H(N)$ and H/N is simple, we have that $C_H(N) = H$. It forces that $H \cong \mathbb{Z}_2 \times \mathrm{A}_7$ or $\mathbb{Z}_2.\mathrm{A}_7$. If $H \cong \mathbb{Z}_2 \times \mathrm{A}_7$, then H has a characteristic subgroup $K \cong \mathrm{A}_7$. The normality of H in M implies that K is also normal in M. Thus, the block graph X_K has order 4. By Proposition 2.1, K is semiregular and $|K| \,\big|\, 12p$. This is impossible because $K \cong \mathrm{A}_7$. If $H \cong \mathbb{Z}_2.\mathrm{A}_7$, then $|H_v| = 168$. However, by Magma [3], $\mathbb{Z}_2.\mathrm{A}_7$ has no subgroups of order 168, a contradiction.

Assume that $X_N \cong \mathcal{C}^1_{78}$ or $\mathcal{C}^2_{78}$. Then $p = 13$. By Atlas [6], $\mathcal{C}^1_{78}$ or $\mathcal{C}^2_{78}$ has a minimal arc-transitive subgroup isomorphic to $\mathrm{PSL}(2, 13)$. Thus, A/N has an arc-transitive subgroup $M/N \cong \mathrm{PSL}(2, 13)$. Since $N \cong \mathbb{Z}_2$, we have that $C_M(N) = M$. By Atlas [6], $\mathrm{Mult}(\mathrm{PSL}(2, 13)) = 2$. It follows that $M \cong \mathrm{SL}(2, 13)$ or $\mathrm{PSL}(2, 13) \times \mathbb{Z}_2$. If $M \cong \mathrm{SL}(2, 13)$, then $M_v \cong \mathbb{Z}_{14}$ by Magma [3]. This is impossible by Proposition 2.2. It follows that $M \cong \mathrm{PSL}(2, 13) \times \mathbb{Z}_2$. Note that M is arc-transitive. Thus, by Construction 2.8 and Lemma 2.9, $X \cong \mathcal{SG}^i_{156}$ with $i = 1, 2, 3, 4, 5$.

Case 2: A has no solvable minimal normal subgroup.

For convenience, we still use N to denote a minimal normal subgroup of A. Since N is non-solvable and minimal normal, we have that $N = T^k$ with T a non-abelian simple group and k an positive integer. It follows that T has at least 3-prime factors. Note that $|N| \bigm| 2^{26}{\cdot}3^5{\cdot}5^2{\cdot}7{\cdot}p$. Thus, T is one of the simple groups listed in Proposition 2.3. By Proposition 2.1, N has at most two orbits on $V(X)$, and hence $|N| = 12p|N_v|$ or $6p|N_v|$.

Assume that $k \geq 2$. Since T is a non-abelian simple group, we have that $2^2 \bigm| |T|$ and $T_v \neq 1$. If $p > 7$, then $p \nmid |T|$ because $p^2 \nmid |N| = |T^k|$. It follows that p divides the order of X_N. By Proposition 2.1, $N = T^k$ is semiregular and hence $T_v = 1$, a contradiction. Thus, $p \leq 7$. Recall that $|T^k| \bigm| 2^{26}{\cdot}3^5{\cdot}5^2{\cdot}7{\cdot}p$ and $6{\cdot}p \bigm| |T^k|$.

Let $p = 5$. Then $7^2 \nmid |T^k|$ and T is a simple $\{2, 3, 5\}$-group. By Proposition 2.3, $T \cong \mathrm{A}_5$ or A_6. The normality of N in A implies that $N_v \trianglelefteq A_v$. If $k \geq 3$, then $5^2 \bigm| |N_v|$ and hence N_v has a subgroup isomorphic to A_7 by Proposition 2.2. It follows that $7 \bigm| |T|$, a contradiction. Thus, $k = 2$. If $T \cong \mathrm{A}_5$, then $N \cong \mathrm{A}_5^2$ and $|N_v| = 60$ or 120. By Atlas [6], for $|N_v| = 60$, we have that $N_v \cong D_{10} \times \mathrm{S}_3$, $\mathrm{A}_4 \times \mathbb{Z}_5$ or A_5, and for $|N_v| = 120$, we have that $N_v \cong \mathrm{A}_5 \times \mathbb{Z}_2$ or $\mathrm{A}_4 \times D_{10}$. However, by Proposition 2.2, A_v has no such normal subgroups, a contradiction. If $T \cong \mathrm{A}_6$, then $N \cong \mathrm{A}_6^2$, and $|N_v| = 2^4{\cdot}3^3{\cdot}5$ or $2^5{\cdot}3^3{\cdot}5$. By Magma [3], for $|N_v| = 2^4{\cdot}3^3{\cdot}5$, we have that $N_v \cong \mathrm{A}_6 \times \mathrm{S}_3$ or $\mathrm{A}_5 \times F_{36}$, and for $|N_v| = 2^5{\cdot}3^3{\cdot}5$, we have that $N_v \cong \mathrm{A}_6 \times \mathrm{A}_4$. Similarly, A_v has no such normal subgroups, a contradiction.

Let $p = 7$. Then $|T^k| \bigm| 2^{26}{\cdot}3^5{\cdot}5^2{\cdot}7^2$ and $6{\cdot}7 \bigm| |T^k|$. It follows that $7 \bigm| |T|$ and $k = 2$. By Proposition 2.3, T is isomorphic to one of the following groups:

$$\mathrm{PSL}(2,7),\ \mathrm{PSL}(2,8),\ \mathrm{A}_7,\ \mathrm{A}_8,\ \mathrm{PSL}(3,4).$$

If $T \cong \mathrm{PSL}(2,7)$, then $|N_v| = 2^5{\cdot}3{\cdot}7$ or $2^4{\cdot}3{\cdot}7$. By Magma [3], for $|N_v| = 2^5{\cdot}3{\cdot}7$, $N_v \cong \mathrm{PSL}(2,7) \times \mathbb{Z}_2^2$ or $\mathrm{PSL}(2,7) \times \mathbb{Z}_4$; for $|N_v| = 2^4{\cdot}3{\cdot}7$, $N_v \cong \mathrm{PSL}(2,7) \times \mathbb{Z}_2$. The normality of N_v in A_v implies that $A_v \cong \mathrm{PSL}(2,7) \times \mathrm{S}_4$ and $N_v \cong \mathrm{PSL}(2,7) \times \mathbb{Z}_2^2$ by Proposition 2.2. It follows that N has two orbits on $V(X)$ and $|A| = 2|A_vN| = 6|N|$. Clearly, $C_A(N) \cap N = 1$. Thus, $|C_A(N)| \bigm| 6$ and $C_A(N)$ is solvable. Since $C_A(N) \trianglelefteq A$, we have that $C_A(N) = 1$ by our assumption. By "N/C-Theorem" (see [15, Chapter I, Theorem 4.5]), $A \cong A/C_A(N) \lesssim \mathrm{Aut}(N)$. However, by Magma [3], $|\mathrm{Aut}(N)| = 8|N|$, which is contrary to the fact that $|A| = 6|N|$. If $T \cong \mathrm{PSL}(2,8)$, then $|N_v| = 2^5{\cdot}3^3{\cdot}7$ or $2^4{\cdot}3^3{\cdot}7$. By Magma [3], the only possible is that $N_v \cong \mathrm{PSL}(2,8) \times \mathrm{S}_3$. However, A_v has no such normal subgroup by Proposition 2.2, a contradiction. If $T \cong \mathrm{A}_7$, then by Magma [3], the only possible is that $|N_v| = 2^5{\cdot}3^3{\cdot}5^2{\cdot}7$ and $N_v \cong \mathrm{A}_7 \times \mathrm{A}_5$. Similarly, A_v has no such normal subgroup, a contradiction. If $T \cong \mathrm{A}_8$ or $\mathrm{PSL}(3,4)$, then $|N_v| = |N|/6p$ or $|N|/12p$. By Magma [3], A_8^2 or $\mathrm{PSL}(3,4)^2$ has no subgroups of such orders, a contradiction.

Thus, $k = 1$ and $N = T$ is a non-abelian simple group. Since $|N| \bigm| 2^{26}{\cdot}3^5{\cdot}5^2{\cdot}7{\cdot}p$, we have that N is isomorphic to one of the simple group listed in Proposition 2.3. By Proposition 2.1, we have that N has at most two orbits on $V(X)$ and $|N_v| = |N|/6p$ or $|N|/12p$.

Subcase 2.1: Suppose that $p = 5$. Then $|N| \,\big|\, 2^{26}{\cdot}3^5{\cdot}5^3{\cdot}7$. By Proposition 2.3, N is isomorphic to one of the following simple groups:

$$\begin{array}{l}\mathrm{A}_5,\ \mathrm{A}_6,\ \mathrm{PSU}(4,2),\ \mathrm{A}_7,\ \mathrm{A}_8,\ \mathrm{A}_9,\ \mathrm{A}_{10}\\ \mathrm{PSL}(3,4),\ \mathrm{PSp}(6,2),\ \mathrm{J}_2,\ \mathrm{P}\Omega^+(8,2).\end{array}$$

Note that $|N_v| = |N|/6{\cdot}5$ or $|N|/12{\cdot}5$.

Let $N \cong \mathrm{A}_5$. Then $N_v \cong \mathbb{Z}_2$ or 1. If $N_v \cong \mathbb{Z}_2$, then since $N_v \trianglelefteq A_v$, we have that $A_v \cong D_{28}$, $F_{42} \times \mathbb{Z}_2$ or $F_{42} \times \mathbb{Z}_6$ by Proposition 2.2. Note that $C_A(N) \cong C_A(N)N/N \le A/N$ and $|A : A_vN| = 2$. Thus, it is easy to deduce that $C_A(N)$ is solvable, which is contrary to our assumption that A has no solvable minimal normal subgroup. If $N_v = 1$, then N is regular on $V(X)$. By "N/C-Theorem" (see [15, Chapter I, Theorem 4.5]), $A/C_A(N) \lesssim \mathrm{Aut}(N)$. Since $\mathrm{Aut}(\mathrm{A}_5) \cong \mathrm{S}_5$, we have that $7 \,\big|\, |C_A(N)|$ and since $C_A(N) \trianglelefteq A$, we have that $C_A(N)$ is non-solvable and has no solvable characteristic subgroup by our assumption. Clearly, $C_A(N)$ is transitive or has two orbits on $V(X)$. Thus, $|A : A_vC_A(N)| = 2$ and $N \cong C_A(N)N/C_A(N) \trianglelefteq A/C_A(N)$. Since $N \cong \mathrm{A}_5$ and $|A/C_A(N) : A_vC_A(N)/C_A(N)| = 2$, we have that $N \le A_vC_A(N) \cong A_v/(C_A(N) \cap A_v)$. It follows that A_v has a normal subgroup isomorphic to A_5, this is impossible by Proposition 2.2.

Let $N \cong \mathrm{A}_6$. Then $|N_v| = 12$ or 6. By Atlas [6], $N_v \cong \mathrm{A}_4$, or S_3. Since $N \trianglelefteq A$, we have that $N_v \trianglelefteq A_v$. By Proposition 2.2, $N_v \cong \mathrm{A}_4$ and $A_v \cong \mathrm{PSL}(2,7) \times \mathrm{S}_4$. It follows that $|A| = 2^4{\cdot}3{\cdot}7{\cdot}|N|$. Similar as above, $C_A(N) \ne 1$ and $C_A(N)$ is no-solvable. Clearly, $C_A(N) \cap N = 1$. Thus, $|C_A(N)| \,\big|\, 2^4{\cdot}3{\cdot}7$. This implies that $C_A(N)$ acting on $V(X)$ has at least five orbits. By Proposition 2.1, $C_A(N)$ is regular and hence solvable, a contradiction.

Let $N \cong \mathrm{PSU}(4,2)$, A_7, A_8, A_9, A_{10}, $\mathrm{PSL}(3,4)$, $\mathrm{PSp}(6,2)$, J_2 or $\mathrm{P}\Omega^+(8,2)$. Then $|N_v| = |N|/30$ or $|N|/60$. However, by Magma [3], these simple groups have no subgroups of such orders, a contradiction.

Subcase 2.2: Suppose that $p = 7$. Then $|N| \,\big|\, 2^{26}{\cdot}3^5{\cdot}5^2{\cdot}7^2$ and $6{\cdot}7 \,\big|\, |N|$ or $12{\cdot}7 \,\big|\, |N|$. By Proposition 2.3, N is isomorphic to one of the following simple groups:

$$\begin{array}{l}\mathrm{PSL}(2,7),\ \mathrm{PSL}(2,8),\ \mathrm{PSU}(3,3),\ \mathrm{A}_7,\ \mathrm{A}_8,\ \mathrm{A}_9,\ \mathrm{A}_{10}\\ \mathrm{PSL}(2,49),\ \mathrm{PSL}(3,4),\ \mathrm{PSp}(6,2),\ \mathrm{J}_2,\ \mathrm{P}\Omega^+(8,2).\end{array}$$

Let $N \cong \mathrm{PSL}(2,7)$. Then $N_v \cong \mathbb{Z}_2$ or $\mathbb{Z}_2^2$. Set $C = C_A(N)$. If $C = 1$, then by "N/C-Theorem", $A \cong A/C \lesssim \mathrm{Aut}(N) \cong \mathrm{PGL}(2,7)$. However, since $p = 7$ and X is arc-transitive, we have that $7^2 \,\big|\, |A|$, a contradiction. Thus, $C \ne 1$. Since $C \trianglelefteq A$, we have that C is non-solvable by our assumption. Note that $C \cong CN/N \trianglelefteq A/N$ and $A_v/N_v \cong A_vN/N \le A/N$ with $|A : A_vN| \le 2$. Thus, A_v is non-solvable. Since $N_v \trianglelefteq A_v$, we have that $N_v \cong \mathbb{Z}_2^2$ and $A_v \cong \mathrm{PSL}(2,7) \times \mathrm{S}_4$ by Proposition 2.2. It follows that $A_v/N_v \cong \mathrm{PSL}(2,7) \times \mathrm{S}_3$ and $C \cong \mathrm{PSL}(2,7)$ because C is non-solvable and has no solvable characteristic subgroup. Thus, $|A/C : NC/C| = 12$ and $N \lesssim A/C \lesssim \mathrm{Aut}(N) \cong \mathrm{PGL}(2,7)$, this is impossible.

Let $N \cong \mathrm{PSL}(2,8)$. Then $|N_v| = 6$ or 12. By Atlas [6], $N_v \cong \mathrm{S}_3$. Since $N \trianglelefteq A$, we have that $N_v \trianglelefteq A_v$. However, by Proposition 2.2, A_v has no normal subgroup isomorphic to S_3, a contradiction.

Let $N \cong \mathrm{PSU}(3,3)$, A_8, A_{10}, $\mathrm{PSL}(2,49)$, $\mathrm{PSL}(3,4)$, $\mathrm{PSp}(6,2)$, J_2 or $\mathrm{P}\Omega^+(8,2)$. Then $|N_v| = |N|/42$ or $|N|/84$. By Magma [3], N has no subgroups of such orders, a contradiction.

Let $N \cong \mathrm{A}_7$. Then $|N_v| = 30$ or 60. By Atlas [6], $N_v \cong \mathrm{A}_5$. The normality of N in A implies that A_v has a normal subgroup isomorphic to A_5. This is impossible by Proposition 2.2.

Let $N \cong \mathrm{A}_9$. Then by Atlas [6], $N_v \cong (\mathrm{A}_6 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_2$. This is also impossible because A_v has no normal subgroup isomorphic to $(\mathrm{A}_6 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_2$.

Subcase 2.3: Suppose that $p > 7$. Then $|N| \bigm| 2^{26}{\cdot}3^5{\cdot}5^2{\cdot}7{\cdot}p$ and $6{\cdot}p \bigm| |N|$ or $12{\cdot}p \bigm| |N|$. By Proposition 2.3, N is isomorphic to one of the following simple groups:

$$\begin{array}{l}
\mathrm{PSL}(2,17),\ \mathrm{PSL}(3,3),\ \mathrm{PSL}(2,11),\ \mathrm{PSL}(2,13),\ \mathrm{PSL}(2,16)\\
\mathrm{PSL}(2,19),\ \mathrm{PSL}(2,25),\ \mathrm{PSL}(2,27),\ \mathrm{PSL}(2,31),\ \mathrm{PSL}(2,81)\\
\mathrm{PSL}(2,127),\ \mathrm{PSU}(3,4),\ \mathrm{PSU}(3,8),\ \mathrm{PSU}(5,2),\ \mathrm{PSp}(4,4),\ \mathrm{M}_{11},\ \mathrm{M}_{12}\\
{}^2F_4(2)',\ \mathrm{A}_{11},\ \mathrm{A}_{12},\ \mathrm{PSL}(2,29),\ \mathrm{PSL}(2,41),\ \mathrm{PSL}(2,71),\ \mathrm{PSL}(2,449)\\
\mathrm{PSL}(2,2^6),\ \mathrm{PSL}(4,4),\ \mathrm{PSL}(5,2),\ \mathrm{PSp}(8,2),\ \mathrm{M}_{22},\ \mathrm{P}\Omega^-(8,2),\ G_2(4).
\end{array}$$

Let $N \cong \mathrm{PSL}(2,17)$. Then by Atlas [6], $N_v \cong \mathrm{A}_4$ or S_4. Since $N_v \trianglelefteq A_v$, we have that $A_v \cong \mathrm{PSL}(2,7) \times \mathrm{S}_4$ by Proposition 2.2. Similar as above, $C_A(N) \cong \mathrm{PSL}(2,7)$. Since $p \nmid |C_A(N)|$, we have that $C_A(N)$ has at least p orbits on $V(X)$, and by Proposition 2.1, $C_A(N)$ is semiregular and hence solvable, a contradiction.

Let $N \cong \mathrm{PSL}(3,3)$. Then $|N_v| = 2^2{\cdot}3^2$ or $2^3{\cdot}3^2$. By Atlas [6], N_v has a characteristic subgroup isomorphic to $\mathbb{Z}_3^2$. The normality of N_v in A_v implies that A_v has a normal subgroup isomorphic to $\mathbb{Z}_3^2$. This is impossible by Proposition 2.2.

Let $N \cong \mathrm{PSL}(2,11)$, $\mathrm{PSL}(2,25)$, M_{11}, M_{12} or A_{12}. Then by Atlas [6], $N_v \cong \mathbb{Z}_5$ or D_{10} for $N \cong \mathrm{PSL}(2,11)$; $N_v \cong \mathbb{Z}_5^2 \rtimes \mathbb{Z}_2$ or $\mathbb{Z}_5^2 \rtimes \mathbb{Z}_4$ for $N \cong \mathrm{PSL}(2,15)$; $N_v \cong \mathrm{A}_5$ or S_5 for $N \cong \mathrm{M}_{11}$; $N_v \cong \mathrm{A}_6 \rtimes \mathbb{Z}_2^2$ or $\mathrm{A}_6 \times \mathbb{Z}_2$ for $N \cong \mathrm{M}_{12}$; $N_v \cong \mathrm{A}_{10}$ or S_{10} for $N \cong \mathrm{A}_{12}$. However, By Proposition 2.2, A_v has no such normal subgroups, a contradiction.

Let $N \cong \mathrm{PSL}(2,13)$. Then by Atlas [6], $N_v \cong \mathbb{Z}_7$ or D_{14}. Since $N_v \trianglelefteq A_v$, we have that A_v is solvable by Proposition 2.2. Note that $C_A(N) \cong C_A(N)N/N \trianglelefteq A/N$ and $|A/N : A_vN/N| \leq 2$. Thus, $C_A(N)$ is solvable. By our assumption, $C_A(N) = 1$ and by "N/C-Theorem", $A \cong A/C \lesssim \mathrm{Aut}(N) \cong \mathrm{PGL}(2,13)$. By Construction 2.8 and Lemma 2.9, $X \cong \mathcal{CG}^i_{156}$ with $i = 1,2,3,4$.

For the remaining simple groups listed in the above, with the calculation of Magma [3] or check the information of maximal subgroups in Atlas [6], we can deduce that all the groups do not have subgroups of order $|N_v| = |N|/6{\cdot}p$ or $|N|/12{\cdot}p$, a contradiction. $\square$

Acknowledgements: This work was supported by the National Natural Science Foundation of China (11301154), and the Innovation Team Funding of Henan University of Science and Technology(2015XTD010).

References

[1] N. Biggs, *Algebraic graph theory,* Second ed., Cambridge University Press, Cambridge, 1993.

[2] J.A. Bondy, U.S.R. Murty, *Graph theory with applications*, Elsevier Science Ltd, New York, 1976.

[3] W. Bosma, C. Cannon, C. Playoust, *The MAGMA algebra system I: The user language*, J. Symbolic Comput., 24 (1997), 235-265.

[4] Y. Cheng, J. Oxley, *On the weakly symmetric graphs of order twice a prime*, J. Combin. Theory B, 42 (1987), 196-211.

[5] M.D.E. Conder, *A complete list of all connected symmetric graphs of order* 2 *to* 30*,* `https://www.math.auckland.ac.nz/conder/symmetricgraphs-orderupto30.txt`.

[6] H.J. Conway, R.T. Curtis, S.P. Norton, R.A. Parker, and R.A Wilson, *Atlas of finite group*, Clarendon Press, Oxford, 1985.

[7] X.G. Fang, C.E. Praeger, *Finite two-arc transitive graphs admitting a Suzuki simple group,* Comm. Algebra, 27 (1999), 3727-3754.

[8] Y.Q. Feng, J.H. Kwak, K.S. Wang, *Classifying cubic symmetric graphs of order* $8p$ *or* $8p^2$, European J. Combin., 26 (2005), 1033-1052.

[9] D. Gorenstein, *Finite simple groups,* Plenum Press, New York, 1982.

[10] S.T. Guo, *Heptavalent symmetric graphs of order* $6p$, Ita. J. Pure Appl. Math., 38 (2017), 741-750.

[11] S.T. Guo, *Heptavalent symmetric graphs of order* $18p$, Submitted.

[12] S.T. Guo, Y.Q. Feng, *A note on pentavalent s-transitive graphs*, Discrete Math., 312 (2012), 2214-2216.

[13] S.T. Guo, Y.T. Li, X.H. Hua, *(G, s)-transitive graphs of valency* 7, Algeb. Colloq., 23 (2016), 493-500.

[14] S.T. Guo, J. Shi, Z.J. Zhang, *Heptavalent symmetric graphs of order* $4p$*,* South Asian J. Math., 1 (2011), 131-136.

[15] B. Huppert, *Eudiche Gruppen I*, Springer-Verlag, Berlin, 1967.

[16] A. Jafarzadeh, A. Iranmanesh, *On simple K_n-groups for $n = 5, 6$, (In Campbell, CM, Quick, MR, Robertson, EF, Smith, GC, eds.)* Groups St. Andrews 2005. London Mathematical Society lecture note series, vol. 2, pp. 668-680, Cambridge University Press , Cambridge, (2007).

[17] B. Ling, C.X. Wu, B. Lou, *Pentavalent symmetric graphs of order* $30p$, Bull. Aust. Math. Soc., 90 (2014), 353-362.

[18] P. Lorimer, *Vertex-transitive graphs: Symmetric graphs of prime valency,* J. Graph Theory, 8 (1984), 55-68.

[19] B.D. Mckay, *Transitive graphs with fewer than 20 vertices*, Math. Comp., 33 (1979), 1101-1121.

[20] R.C. Miller, *The trivalent symmetric graphs of girth at most six*, J. Combin. Theory B, 10 (1971), 163-182.

[21] J. Pan, B. Lou, C. Liu, *Arc-transitive pentavalent graphs of order* $4pq$, Electron. J. Comb., 20 (2013), 1215-1230.

[22] J.M. Pan, X. Yu, *Pentavalent Symmetric Graphs of Order Twice a Prime Square*, Algeb. Colloq., 22 (2015), 383-394.

[23] P. Potočnik, *A list of* 4*-valent* 2*-arc-transitive graphs and finite faithful amalgams of index* $(4, 2)$, European J. Combin., 30 (2009), 1323-1336.

[24] D.J. Robinson, *A Course in the theory of groups*, Springer-Verlag, New York, 1982.

[25] B.O. Sabidussi, *Vertex-transitive graphs*, Monatsh Math., 68 (1964), 426-438.

[26] W.J. Shi, *On simple K4-groups*, Chinese Science Bull, 36 (1991), 1281-1283 (in Chinese).

[27] H. Wielandt, *Finite permutation groups*, Academic Press, New York, 1964.

[28] D.W. Yang, Y.Q. Feng, *Pentavalent symmetric graphs of order* $2p^3$, J. Korean Math. Soc., 18 (2016), 1-18.

[29] D.W. Yang, Y.Q. Feng, J.L. Du, *Pentavalent symmetric graphs of order* $2pqr$, Discrete Math., 339 (2016), 522-532.

[30] J.X. Zhou, *Tetravalent s-transitive graphs of order* $4p$, Discrete Math., 309 (2009), 6081-6086.

[31] J.X. Zhou, Y.Q. Feng, *Tetravalent s-transitive graphs of order twice a prime power*, J. Aust. Math. Soc., 88 (2010), 277-288.

Accepted: 1.12.2017